

On decomposition numbers with Jantzen filtration of cyclotomic q -Schur algebras

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ABSTRACT. Let $\mathcal{S}(\Lambda)$ be the cyclotomic q -Schur algebra associated to the Ariki-Koike algebra $\mathcal{H}_{n,r}$, introduced by Dipper-James-Mathas. In this paper, we consider v -decomposition numbers of $\mathcal{S}(\Lambda)$, namely decomposition numbers with respect to the Jantzen filtrations of Weyl modules. We prove, as a v -analogue of the result obtained by Shoji-Wada, a product formula for v -decomposition numbers of $\mathcal{S}(\Lambda)$, which asserts that certain v -decomposition numbers are expressed as a product of v -decomposition numbers for various cyclotomic q -Schur algebras associated to Ariki-koike algebras \mathcal{H}_{n_i, r_i} of smaller rank. Moreover we prove a similar formula for v -decomposition numbers of $\mathcal{H}_{n,r}$ by using a Schur functor.

0. INTRODUCTION

Let $\mathcal{H} = \mathcal{H}_{n,r}$ be the Ariki-Koike algebra over an integral domain R associated to the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$. Dipper, James and Mathas [DJM] introduced the cyclotomic q -Schur algebra $\mathcal{S}(\Lambda)$ associated to the Ariki-Koike algebra \mathcal{H} , and they showed that \mathcal{H} and $\mathcal{S}(\Lambda)$ are cellular algebras in the sence of Graham and Lehrer [GL], by constructing the cellular basis respectively. It is a fundamental problem for the representation theory to determine the decomposition numbers of \mathcal{H} and $\mathcal{S}(\Lambda)$. It is well-known that the decomposition matrix of \mathcal{H} coincides with the submatrix of that of $\mathcal{S}(\Lambda)$ by the Schur functor.

In the case where \mathcal{H} is the Iwahori-Hecke algebra \mathcal{H}_n of type A, Lascoux, Leclerc and Thibon [LLT] conjectured that the decomposition numbers of \mathcal{H}_n can be described by using the canonical basis of a certain irreducible $U_v(\widehat{\mathfrak{sl}}_e)$ -module, and gave the algorithm to compute this canonical basis. The cojecture has been solved by Ariki [A1], by extending to the case of Ariki-Koike algebras.

In the case of the q -Schur algebra associated to \mathcal{H}_n , Leclerc and Thibon [LT] conjectured that the decomposition matrix coincides with the transition matrix between the canonical basis and the standard basis of the Fock space of level 1 equipped with the $U_v(\widehat{\mathfrak{sl}}_e)$ -module structure, and gave the algorithm to compute the transition matrix. This conjecture has been solved by Varagnolo and Vasserot in [VV].

More generally, in the case of the cyclotomic q -Schur algebra \mathcal{S} , Yvonne [Y] has conjectured that the decomposition matrix coincides with the transition matrix between the canonical basis and the standard basis of the higher-level Fock space. This canonical basis was constructed by Uglov [U] and the algorithm to compute

the transition matrix was also given there. Yvonne's conjecture is still open. We remark that Ariki's theorem, Varagnolo-Vasserot's theorem and Yvonne's conjecture are concerned with the situation where R is a complex number field and parameters are roots of unity.

In order to study the decomposition numbers of \mathcal{S} , we constructed in [SW] some subalgebras $\mathcal{S}^{\mathbf{p}}$ of $\mathcal{S}(\Lambda)$ and their quotients $\overline{\mathcal{S}}^{\mathbf{p}}$, and showed that $\mathcal{S}^{\mathbf{p}}$ is a standardly based algebra in the sense of Du and Rui [DR], and that $\overline{\mathcal{S}}^{\mathbf{p}}$ is a cellular algebra. Hence, one can consider the decomposition numbers of $\mathcal{S}^{\mathbf{p}}$ and $\overline{\mathcal{S}}^{\mathbf{p}}$ also. We denote the decomposition numbers of \mathcal{S} , $\mathcal{S}^{\mathbf{p}}$ and $\overline{\mathcal{S}}^{\mathbf{p}}$ by $d_{\lambda\mu}$, $d_{\lambda\mu}^{(\lambda,0)}$ and $\bar{d}_{\lambda\mu}$ respectively, where $d_{\lambda\mu}$ is a decomposition number of the irreducible module L^μ in the Weyl module W^λ of \mathcal{S} for r -partitions λ, μ , and $d_{\lambda\mu}^{(\lambda,0)}$, $\bar{d}_{\lambda\mu}$ are defined similarly for $\mathcal{S}^{\mathbf{p}}$ and $\overline{\mathcal{S}}^{\mathbf{p}}$ (see Section 1 for details). It is proved in [SW, Theorem 3.13] that

$$(1) \quad \bar{d}_{\lambda\mu} = d_{\lambda\mu}^{(\lambda,0)} = d_{\lambda\mu}$$

whenever λ, μ satisfy a certain condition $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$. Moreover for such λ, μ , the product formula for $\bar{d}_{\lambda\mu}$,

$$(2) \quad \bar{d}_{\lambda\mu} = \prod_{k=1}^g d_{\lambda^{[k]}\mu^{[k]}},$$

was proved in [SW, Theorem 4.17], where $d_{\lambda^{[k]}\mu^{[k]}}$ for $k = 1, \dots, g$ is the decomposition number of the cyclotomic q -Schur algebra associated to a certain Ariki-Koike algebra \mathcal{H}_{n_k, r_k} .

Related to the above conjectures on Fock spaces, Leclerc-Thibon and Yvonne give a more precise conjecture concerning the v -decomposition numbers defined by using Jantzen filtrations of Weyl modules. (For definition of v -decomposition numbers, see §2.) We remark that decomposition numbers coincide with v -decomposition numbers at $v = 1$. Thus we regard v -decomposition numbers as a v -analogue of decomposition numbers. The conjecture for v -decomposition numbers has been still open even in the case of the q -Schur algebra of type A.

In this paper, we show that similar formula as (1) and (2) also hold for v -decomposition numbers. We denote the v -decomposition numbers of $\mathcal{S}(\Lambda)$, $\mathcal{S}^{\mathbf{p}}(\Lambda)$ and $\overline{\mathcal{S}}^{\mathbf{p}}(\Lambda)$ by $d_{\lambda\mu}(v)$, $d_{\lambda\mu}^{(\lambda,0)}(v)$ and $\bar{d}_{\lambda\mu}(v)$ respectively. Then for r -partitions λ, μ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have (Theorem 2.8)

$$\bar{d}_{\lambda\mu}(v) = d_{\lambda\mu}^{(\lambda,0)}(v) = d_{\lambda\mu}(v),$$

and (Theorem 2.14)

$$d_{\lambda\mu}(v) = \bar{d}_{\lambda\mu}(v) = \prod_{k=1}^g d_{\lambda^{[k]}\mu^{[k]}}(v),$$

where $d_{\lambda^{[k]}\mu^{[k]}}(v)$ is the v -decomposition number of the cyclotomic q -Schur algebra appeared in (2).

We note that our result is a v -analogue of (1),(2), and it reduces to them by taking $v \mapsto 1$. Moreover, for a certain v -decomposition number $d_{\lambda\mu}^{\mathcal{H}}(v)$ of the Ariki-Koike algebra, we also have the following product formula (Theorem 3.5).

$$d_{\lambda\mu}^{\mathcal{H}}(v) = \prod_{k=1}^g d_{\lambda^{[k]}\mu^{[k]}}^{\mathcal{H}}(v),$$

where $d_{\lambda^{[k]}\mu^{[k]}}^{\mathcal{H}}(v)$ is the v -decomposition number of the certain Ariki-Koike algebra \mathcal{H}_{n_k, r_k} .

We remark that our results hold for any parameters and any modular system, even for the case where the base field has non-zero characteristic, though Yvonne's conjecture is formulated under certain restrictions for parameters and modular systems.

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1. A REVIEW OF KNOWN RESULTS

1.1. Throughtout the paper, we follow the notation in [SW]. Here we review some of them. We fix positive integers r, n and an r -tuple $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}_{>0}^r$. A composition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a finite sequence of non-negative integers, and $|\lambda| = \sum_i \lambda_i$ is called the size of λ . If $\lambda_l \neq 0$ and $\lambda_k = 0$ for any $k > l$, then l is called the length of λ . If the composition λ is a weakly decreasing sequence, λ is called a partition. An r -tuple $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ of compositions is called the r -composition, and size $|\mu|$ of μ is defined by $\sum_{i=1}^r |\mu^{(i)}|$. In particular, if all $\mu^{(i)}$ are partitions, μ is called an r -partition. We denote by $\Lambda = \tilde{\mathcal{P}}_{n,r}(\mathbf{m})$ the set of r -compositions $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ such that $|\mu| = n$ and that the length of $\mu^{(k)}$ is smaller than m_k for $k = 1, \dots, r$. We define $\Lambda^+ = \mathcal{P}_{n,r}(\mathbf{m})$ as the subset of Λ consisting of r -partitions.

We define the partial order, the so-called “dominance order”, on Λ by $\mu \succeq \nu$ if and only if

$$\sum_{i=1}^l |\mu^{(i)}| + \sum_{j=1}^k \mu_j^{(l)} \geq \sum_{i=1}^l |\nu^{(i)}| + \sum_{j=1}^k \nu_j^{(l)}$$

for any $1 \leq l \leq r, 1 \leq k \leq m_l$. If $\mu \succeq \nu$ and $\mu \neq \nu$, we write it as $\mu \triangleright \nu$.

For $\lambda \in \Lambda^+$, we denote by $\text{Std}(\lambda)$ the set of standard tableau of shape λ . For $\lambda \in \Lambda^+$ and $\mu \in \Lambda$, we denote by $\mathcal{T}_0(\lambda, \mu)$ the set of semistandard λ -tableau of type μ . Moreover we set $\mathcal{T}_0(\lambda) = \cup_{\mu \in \Lambda} \mathcal{T}_0(\lambda, \mu)$. For definitions of standard tableau and semistandard tableau, see [SW] or [DJM].

1.2. Let $\mathcal{H} = \mathcal{H}_{n,r}$ be the Ariki-Koike algebra over an integral domain R with parameters q, Q_1, \dots, Q_r with defining relations in [SW, §1.1]. It is known by [DJM] that \mathcal{H} has a structure of the cellular algebra with a cellular basis $\{m_{\mathbf{s}\mathbf{t}} | \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda) \text{ for some } \lambda \in \Lambda^+\}$. Then the general theory of a cellular algebra by [GL] implies the following results. There exists an anti-automorphism $h \mapsto h^*$ of \mathcal{H} such

that $m_{\mathfrak{s}\mathfrak{t}}^* = m_{\mathfrak{t}\mathfrak{s}}$. For $\lambda \in \Lambda^+$, let $\mathcal{H}^{\vee\lambda}$ be the R -submodule of \mathcal{H} spanned by $m_{\mathfrak{s}\mathfrak{t}}$, where $\mathfrak{s}, \mathfrak{t} \in \text{Std}(\mu)$ for some $\mu \in \Lambda^+$ such that $\mu \triangleright \lambda$. Then $\mathcal{H}^{\vee\lambda}$ is an ideal of \mathcal{H} . One can construct the standard (right) \mathcal{H} -module S^λ , called a Specht module, with the R -free basis $\{m_{\mathfrak{t}} \mid \mathfrak{t} \in \text{Std}(\lambda)\}$. We define the bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ on S^λ by

$$\langle m_{\mathfrak{s}}, m_{\mathfrak{t}} \rangle_{\mathcal{H}} m_{\mathfrak{u}\mathfrak{v}} \equiv m_{\mathfrak{u}\mathfrak{s}} m_{\mathfrak{t}\mathfrak{v}} \pmod{\mathcal{H}^{\vee\lambda}} \quad (\mathfrak{s}, \mathfrak{t} \in \text{Std}(\lambda)),$$

where $\mathfrak{u}, \mathfrak{v} \in \text{Std}(\lambda)$, and the scalar $\langle m_{\mathfrak{s}}, m_{\mathfrak{t}} \rangle_{\mathcal{H}}$ does not depend on the choice of $\mathfrak{u}, \mathfrak{v} \in \text{Std}(\lambda)$. The bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is associative, namely we have

$$(1.2.1) \quad \langle xh, y \rangle_{\mathcal{H}} = \langle x, yh^* \rangle_{\mathcal{H}} \quad \text{for } x, y \in S^\lambda, h \in \mathcal{H}.$$

Let $\text{rad } S^\lambda = \{x \in S^\lambda \mid \langle x, y \rangle_{\mathcal{H}} = 0 \text{ for any } y \in S^\lambda\}$. Then $\text{rad } S^\lambda$ is the \mathcal{H} -submodule of S^λ by the associativity of the bilinear form. Put $D^\lambda = S^\lambda / \text{rad } S^\lambda$. Assume that R is a field. Then D^λ is an absolutely irreducible module or zero, and the set $\{D^\lambda \mid \lambda \in \Lambda^+ \text{ such that } D^\lambda \neq 0\}$ gives a complete set of non-isomorphic irreducible \mathcal{H} -modules.

1.3. Let $\mathcal{S} = \mathcal{S}(\Lambda)$ be the cyclotomic q -Schur algebra introduced by [DJM], associated to the Ariki-Koike algebra \mathcal{H} with respect to the set Λ . It is known by [DJM] that \mathcal{S} is a cellular algebra with a cellular basis $\{\varphi_{ST} \mid S, T \in \mathcal{T}_0(\lambda) \text{ for some } \lambda \in \Lambda^+\}$. Again by the general theory of a cellular algebra, the following results hold. There exists the anti-automorphism $x \mapsto x^*$ of \mathcal{S} such that $\varphi_{ST}^* = \varphi_{TS}$. For $\lambda \in \Lambda^+$, let $\mathcal{S}^{\vee\lambda}$ be the R -submodule spanned by φ_{ST} , where $S, T \in \mathcal{T}_0(\mu)$ for some $\mu \in \Lambda^+$ such that $\mu \triangleright \lambda$. Then $\mathcal{S}^{\vee\lambda}$ is an ideal of \mathcal{S} . One can construct the standard (right) \mathcal{S} -module W^λ ($\lambda \in \Lambda^+$), called a Weyl module, with the R -free basis $\{\varphi_T \mid T \in \mathcal{T}_0(\lambda)\}$. We define a bilinear form $\langle \cdot, \cdot \rangle$ on W^λ by

$$\langle \varphi_S, \varphi_T \rangle \varphi_{UV} \equiv \varphi_{US} \varphi_{TV} \pmod{\mathcal{S}^{\vee\lambda}} \quad (S, T \in \mathcal{T}_0(\lambda)),$$

where $U, V \in \mathcal{T}_0(\lambda)$, and the scalar $\langle \varphi_S, \varphi_T \rangle$ does not depend on a choice of $U, V \in \mathcal{T}_0(\lambda)$. The bilinear form $\langle \cdot, \cdot \rangle$ is associative, namely we have

$$(1.3.1) \quad \langle x\varphi, y \rangle = \langle x, y\varphi^* \rangle \quad \text{for } x, y \in W^\lambda, \varphi \in \mathcal{S}.$$

Let $\text{rad } W^\lambda = \{x \in W^\lambda \mid \langle x, y \rangle = 0 \text{ for any } y \in W^\lambda\}$. Then $\text{rad } W^\lambda$ is the \mathcal{S} -submodule of W^λ . Put $L^\lambda = W^\lambda / \text{rad } W^\lambda$. Then it is known by [DJM] that $L^\lambda \neq 0$ for any $\lambda \in \Lambda^+$. Assume that R is a field. Then L^λ is an absolutely irreducible module, and the set $\{L^\lambda \mid \lambda \in \Lambda^+\}$ gives a complete set of non-isomorphic irreducible \mathcal{S} -modules.

1.4. We recall some definitions and results in [SW]. We fix a positive integer $g \leq r$ and $\mathbf{p} = (r_1, \dots, r_g) \in \mathbb{Z}_{>0}^g$ such that $r_1 + \dots + r_g = r$, and set $p_1 = 0, p_i = \sum_{j=1}^{i-1} r_j$ for $i = 2, \dots, g$. For $\mu = (\mu^{(1)}, \dots, \mu^{(r)}) \in \Lambda$, we define $\alpha_{\mathbf{p}}(\mu) = (n_1, \dots, n_g)$ and $\mathbf{a}_{\mathbf{p}}(\mu) = (a_1, \dots, a_g)$, where $n_k = \sum_{i=1}^{r_k} |\mu^{(p_k+i)}|$ and $a_k = \sum_{i=1}^{k-1} n_i$ for $k = 1, \dots, g$ with $a_1 = 0$. We define a partial order on $\mathbb{Z}_{>0}^g$ by $\mathbf{a} = (a_1, \dots, a_g) \geq \mathbf{b} = (b_1, \dots, b_g)$

if $a_i \geq b_i$ for any $i = 1, \dots, g$ and we write $\mathbf{a} > \mathbf{b}$ if $\mathbf{a} \geq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$. Later we consider the partial order on $\{\mathbf{a}_{\mathbf{p}}(\mu) \mid \mu \in \Lambda\}$ by this order.

For $\lambda \in \Lambda^+$ and $\mu \in \Lambda$, we set $\mathcal{T}_0^{\mathbf{p}}(\lambda, \mu) = \mathcal{T}_0(\lambda, \mu)$ if $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, and is empty otherwise. Moreover we set $\mathcal{T}_0^{\mathbf{p}}(\lambda) = \bigcup_{\mu \in \Lambda} \mathcal{T}_0^{\mathbf{p}}(\lambda, \mu)$. We set

$$\Sigma^{\mathbf{p}} = (\Lambda^+ \times \{0, 1\}) \setminus \left\{ (\lambda, 1) \in \Lambda^+ \times \{0, 1\} \mid \mathcal{T}_0(\lambda, \mu) = \emptyset \text{ for any } \mu \in \Lambda \text{ such that } \mathbf{a}_{\mathbf{p}}(\lambda) > \mathbf{a}_{\mathbf{p}}(\mu) \right\},$$

and define a partial order \geq on $\Sigma^{\mathbf{p}}$ by $(\lambda_1, \varepsilon_1) > (\lambda_2, \varepsilon_2)$ if $\lambda_1 \triangleright \lambda_2$ or if $\lambda_1 = \lambda_2$ and $\varepsilon_1 > \varepsilon_2$. For $\eta = (\lambda, \varepsilon) \in \Sigma^{\mathbf{p}}$, we set

$$I(\eta) = \begin{cases} \mathcal{T}_0^{\mathbf{p}}(\lambda) & \text{if } \varepsilon = 0, \\ \bigcup_{\substack{\mu \in \Lambda \\ \mathbf{a}_{\mathbf{p}}(\lambda) > \mathbf{a}_{\mathbf{p}}(\mu)}} \mathcal{T}_0(\lambda, \mu) & \text{if } \varepsilon = 1, \end{cases}$$

$$J(\eta) = \begin{cases} \mathcal{T}_0^{\mathbf{p}}(\lambda) & \text{if } \varepsilon = 0, \\ \mathcal{T}_0(\lambda) & \text{if } \varepsilon = 1, \end{cases}$$

$$\mathcal{C}^{\mathbf{p}}(\eta) = \{\varphi_{ST} \mid (S, T) \in I(\eta) \times J(\eta)\} \quad \text{for } \eta \in \Sigma^{\mathbf{p}},$$

and

$$\mathcal{C}^{\mathbf{p}} = \bigcup_{\eta \in \Sigma^{\mathbf{p}}} \mathcal{C}^{\mathbf{p}}(\eta).$$

Let $\mathcal{S}^{\mathbf{p}} = \mathcal{S}^{\mathbf{p}}(\Lambda)$ be the R -submodule of $\mathcal{S}(\Lambda)$ spanned by $\mathcal{C}^{\mathbf{p}}$. We also define $(\mathcal{S}^{\mathbf{p}})^{\vee \eta}$ as the R -submodule of $\mathcal{S}^{\mathbf{p}}$ spanned by

$$\{\varphi_{UV} \mid (U, V) \in I(\eta') \times J(\eta') \text{ for some } \eta' \in \Sigma^{\mathbf{p}} \text{ such that } \eta' > \eta\}.$$

It is known by [SW, Theorem 2.6] that $\mathcal{S}^{\mathbf{p}}$ is a standardly based algebra with the standard basis $\mathcal{C}^{\mathbf{p}}$ in the sence of [DR].

By the general theory of standardly based algebra due to [DR], we have the following results. For $\eta \in \Sigma^{\mathbf{p}}$, one can consider the standard left $\mathcal{S}^{\mathbf{p}}$ -modules ${}^{\diamond}Z^{\eta}$ with the basis $\{\varphi_T^{\eta} \mid T \in I(\eta)\}$ and the standard right $\mathcal{S}^{\mathbf{p}}$ -module Z^{η} with the basis $\{\varphi_T^{\eta} \mid T \in J(\eta)\}$. We call them Weyl modules of $\mathcal{S}^{\mathbf{p}}$. We define the bilinear form $\beta_{\eta} : {}^{\diamond}Z^{\eta} \times Z^{\eta} \rightarrow R$ by

$$\beta_{\eta}(\varphi_S^{\eta}, \varphi_T^{\eta}) \varphi_{UV} \equiv \varphi_{UT} \varphi_{SV} \pmod{(\mathcal{S}^{\mathbf{p}})^{\vee \eta}} \quad (S \in I(\eta), T \in J(\eta)),$$

where β_{η} is determined independent of the choice of $U \in I(\eta)$ and $V \in J(\eta)$. The bilinear form β_{η} is associative, namely we have

$$(1.4.1) \quad \beta_{\eta}(\varphi x, y) = \beta_{\eta}(x, y \varphi) \quad \text{for } x \in {}^{\diamond}Z^{\eta}, y \in Z^{\eta}, \varphi \in \mathcal{S}^{\mathbf{p}}.$$

Let $\text{rad } Z^\eta = \{x \in Z^\eta \mid \beta_\eta(y, x) = 0 \text{ for any } y \in {}^\diamond Z^\eta\}$. Then $\text{rad } Z^\eta$ is a $\mathcal{S}^\mathbf{P}$ -submodule of Z^η by associativity of β_η . Put $L^\eta = Z^\eta / \text{rad } Z^\eta$. Assume that R is a field. Then L^η is an absolutely irreducible module or zero, and the set $\{L^\eta \mid \eta \in \Sigma^\mathbf{P} \text{ such that } \beta_\eta \neq 0\}$ is a complete set of non-isomorphic irreducible (right) $\mathcal{S}^\mathbf{P}$ -modules.

Later we shall only consider the Weyl modules Z^η and irreducible modules L^η of $\mathcal{S}^\mathbf{P}$ for η of the form $(\lambda, 0)$. Note that the composition factors of $Z^{(\lambda, 0)}$ are isomorphic to $L^{(\mu, 0)}$ for some $\mu \in \Lambda^+$ by [SW, Proposition 3.3 (i)].

1.5. Let $\widehat{\mathcal{S}^\mathbf{P}}$ be the R -submodule of $\mathcal{S}^\mathbf{P}$ spanned by

$$\mathcal{C}^\mathbf{P} \setminus \{\varphi_{ST} \mid S, T \in \mathcal{T}_0^\mathbf{P}(\lambda) \text{ for some } \lambda \in \Lambda^+\}.$$

It is known by [SW] that $\widehat{\mathcal{S}^\mathbf{P}}$ is a two-sided ideal of $\mathcal{S}^\mathbf{P}$. Thus, we can define the quotient algebra

$$\overline{\mathcal{S}^\mathbf{P}} = \mathcal{S}^\mathbf{P} / \widehat{\mathcal{S}^\mathbf{P}}.$$

We denote by $\overline{\varphi}$ the image of $\varphi \in \mathcal{S}^\mathbf{P}$ under the natural surjection $\pi : \mathcal{S}^\mathbf{P} \rightarrow \overline{\mathcal{S}^\mathbf{P}}$, and set

$$\overline{\mathcal{C}^\mathbf{P}} = \{\overline{\varphi}_{ST} \mid S, T \in \mathcal{T}_0^\mathbf{P}(\lambda) \text{ for some } \lambda \in \Lambda^+\}.$$

Then $\overline{\mathcal{C}^\mathbf{P}}$ is a free R -basis of $\overline{\mathcal{S}^\mathbf{P}}$. By [SW, Theorem 2.13], $\overline{\mathcal{S}^\mathbf{P}}$ turns out to be a cellular algebra with the cellular basis $\overline{\mathcal{C}^\mathbf{P}}$. Hence by the general theory of cellular algebra, the following results hold. For $\lambda \in \Lambda^+$, we can consider the standard (right) $\overline{\mathcal{S}^\mathbf{P}}$ -module \overline{Z}^λ with the free R -basis $\{\overline{\varphi}_T \mid T \in \mathcal{T}_0^\mathbf{P}(\lambda)\}$. We call it a Weyl module of $\overline{\mathcal{S}^\mathbf{P}}$. We define the bilinear form $\langle \cdot, \cdot \rangle_\mathbf{P} : \overline{Z}^\lambda \times \overline{Z}^\lambda \rightarrow R$ by

$$\langle \overline{\varphi}_S, \overline{\varphi}_T \rangle_\mathbf{P} \overline{\varphi}_{UV} \equiv \overline{\varphi}_{US} \overline{\varphi}_{TV} \pmod{(\overline{\mathcal{S}^\mathbf{P}})^{\vee\lambda}} \quad (S, T \in \mathcal{T}_0^\mathbf{P}(\lambda)),$$

where $\langle \cdot, \cdot \rangle_\mathbf{P}$ is determined independent of the choice $U, V \in \mathcal{T}_0^\mathbf{P}(\lambda)$, and $(\overline{\mathcal{S}^\mathbf{P}})^{\vee\lambda}$ is the R -submodule of $\overline{\mathcal{S}^\mathbf{P}}$ spanned by

$$\{\overline{\varphi}_{ST} \mid S, T \in \mathcal{T}_0^\mathbf{P}(\lambda') \text{ for some } \lambda' \in \Lambda^+ \text{ such that } \lambda' \triangleright \lambda\}.$$

The bilinear form $\langle \cdot, \cdot \rangle_\mathbf{P}$ is associative, namely we have

$$(1.5.1) \quad \langle \overline{x} \overline{\varphi}, \overline{y} \rangle_\mathbf{P} = \langle \overline{x}, \overline{y} \overline{\varphi}^* \rangle_\mathbf{P} \quad \text{for any } \overline{x}, \overline{y} \in \overline{Z}^\lambda, \overline{\varphi} \in \overline{\mathcal{S}^\mathbf{P}}.$$

Let $\text{rad } \overline{Z}^\lambda = \{\overline{x} \in \overline{Z}^\lambda \mid \langle \overline{x}, \overline{y} \rangle_\mathbf{P} = 0 \text{ for any } \overline{y} \in \overline{Z}^\lambda\}$, then $\text{rad } \overline{Z}^\lambda$ is an $\overline{\mathcal{S}^\mathbf{P}}$ -submodule of \overline{Z}^λ . Put $\overline{L}^\lambda = \overline{Z}^\lambda / \text{rad } \overline{Z}^\lambda$. Assume that R is a field. Then \overline{L}^λ is an absolutely irreducible module, and the set $\{\overline{L}^\lambda \mid \lambda \in \Lambda^+\}$ is a complete set of non-isomorphic irreducible (right) $\overline{\mathcal{S}^\mathbf{P}}$ -modules.

1.6. Assuming that R is a field, we set, for $\lambda, \mu \in \Lambda^+$,

$$\begin{aligned} d_{\lambda\mu} &= [W^\lambda : L^\mu], \\ d_{\lambda\mu}^{(\lambda,0)} &= [Z^{(\lambda,0)} : L^{(\mu,0)}], \\ \bar{d}_{\lambda\mu} &= [\bar{Z}^\lambda : \bar{L}^\mu], \end{aligned}$$

where $[W^\lambda : L^\mu]$ is the decomposition number of L^μ in W^λ , and similarly for $\mathcal{S}^{\mathbf{p}}$ and $\bar{\mathcal{S}}^{\mathbf{p}}$. The following theorem was proved in [SW].

Theorem 1.7. [SW, Theorem 3.13] *Assume that R is a field. For $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have*

$$\bar{d}_{\lambda\mu} = d_{\lambda\mu}^{(\lambda,0)} = d_{\lambda\mu}$$

1.8. For $\mu = (\mu^{(1)}, \dots, \mu^{(r)}) \in \Lambda$, we write it in the form $\mu = (\mu^{[1]}, \dots, \mu^{[g]})$, where $\mu^{[i]} = (\mu^{(p_i+1)}, \dots, \mu^{(p_i+r_i)})$. According to the expression of μ as above, $T = (T^{(1)}, \dots, T^{(r)}) \in \mathcal{T}_0(\lambda)$ can be expressed as $T = (T^{[1]}, \dots, T^{[g]})$ with $T^{[i]} = (T^{(p_i+1)}, \dots, T^{(p_i+r_i)})$. By [SW, Lemma 4.3 (iii)], we have a bijection $\mathcal{T}_0^{\mathbf{p}}(\lambda, \mu) \simeq \mathcal{T}_0(\lambda^{[1]}, \mu^{[1]}) \times \dots \times \mathcal{T}_0(\lambda^{[g]}, \mu^{[g]})$ given by the map $T \mapsto (T^{[1]}, \dots, T^{[g]})$. Thus we have a bijection $\mathcal{T}_0^{\mathbf{p}}(\lambda) \simeq \mathcal{T}_0(\lambda^{[1]}) \times \dots \times \mathcal{T}_0(\lambda^{[g]})$.

We write $\mathbf{m} = (m_1, \dots, m_r)$ in the form $\mathbf{m} = (\mathbf{m}^{[1]}, \dots, \mathbf{m}^{[g]})$, where $\mathbf{m}^{[k]} = (m_{p_k+1}, \dots, m_{p_k+r_k})$. For each $n_k \in \mathbb{Z}_{\geq 0}$, put $\Lambda_{n_k} = \tilde{\mathcal{P}}_{n_k, r_k}(\mathbf{m}^{[k]})$, and $\Lambda_{n_k}^+ = \mathcal{P}_{n_k, r_k}(\mathbf{m}^{[k]})$. (Λ_{n_k} or $\Lambda_{n_k}^+$ is regarded as the empty set if $n_k = 0$.) Let $\mathcal{S}(\Lambda_{n_k})$ be the cyclotomic q -Schur algebra associated to the Ariki-Koike algebra \mathcal{H}_{n_k, r_k} with parameters $q, Q_{p_k+1}, \dots, Q_{p_k+r_k}$. Let $\Delta_{n,g}$ be the set of $(n_1, \dots, n_g) \in \mathbb{Z}_{\geq 0}^g$ such that $n_1 + \dots + n_g = n$. Then we have the following decomposition theorem of $\bar{\mathcal{S}}^{\mathbf{p}}$ by [SW, Theorem 4.15].

$$(1.8.1) \quad \bar{\mathcal{S}}^{\mathbf{p}}(\Lambda) \cong \bigoplus_{(n_1, \dots, n_g) \in \Delta_{n,g}} \mathcal{S}(\Lambda_{n_1}) \otimes \dots \otimes \mathcal{S}(\Lambda_{n_g}) \quad \text{as } R\text{-algebra,}$$

under the isomorphism given by

$$(1.8.2) \quad \bar{\varphi}_{ST} \mapsto \varphi_{S^{[1]}T^{[1]}} \otimes \dots \otimes \varphi_{S^{[g]}T^{[g]}} \quad \text{for } S, T \in \mathcal{T}_0^{\mathbf{p}}(\lambda).$$

Assuming that R is a field, for $\lambda^{[k]} \in \Lambda_{n_k}$, let $W^{\lambda^{[k]}}$ be the Weyl module of $\mathcal{S}(\Lambda_{n_k})$, and $L^{\lambda^{[k]}} = W^{\lambda^{[k]}} / \text{rad } W^{\lambda^{[k]}}$ be the irreducible module. By [SW, Corollary 4.16], the following properties hold. Under the isomorphism in (1.8.1), we have, for

$\lambda, \mu \in \Lambda^+$,

$$(1.8.3) \quad \overline{Z}^\lambda \cong W^{\lambda^{[1]}} \otimes \cdots \otimes W^{\lambda^{[g]}},$$

$$(1.8.4) \quad \overline{L}^\mu \cong L^{\mu^{[1]}} \otimes \cdots \otimes L^{\mu^{[g]}},$$

$$(1.8.5) \quad [\overline{Z}^\lambda : \overline{L}^\mu] = \begin{cases} \prod_{k=1}^g [W^{\lambda^{[k]}} : L^{\mu^{[k]}}] & \text{if } \alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu) \\ 0 & \text{otherwise} \end{cases}.$$

Under the isomorphism in (1.8.3), a bilinear form $\langle \cdot, \cdot \rangle_{\mathbf{p}}$ on \overline{Z}^λ decomposes to a product of bilinear forms on $W^{\lambda^{[k]}}$ for $k = 1, \dots, g$, namely we have the following lemma.

Lemma 1.9. *For $S, T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)$, we have*

$$\langle \overline{\varphi}_S, \overline{\varphi}_T \rangle_{\mathbf{p}} = \langle \varphi_{S^{[1]}}, \varphi_{T^{[1]}} \rangle \cdots \langle \varphi_{S^{[g]}}, \varphi_{T^{[g]}} \rangle,$$

where $\langle \varphi_{S^{[k]}}, \varphi_{T^{[k]}} \rangle$ denotes the bilinear form on $W^{\lambda^{[k]}}$ for $k = 1, \dots, g$.

Proof. Fix $U, V \in \mathcal{T}_0^{\mathbf{p}}(\lambda)$. Then by (1.8.2) and the definition of the bilinear form on $W^{\lambda^{[k]}}$, we have

$$\begin{aligned} \overline{\varphi}_{US} \overline{\varphi}_{TV} &= (\varphi_{U^{[1]}S^{[1]}} \otimes \cdots \otimes \varphi_{U^{[g]}S^{[g]}})(\varphi_{T^{[1]}V^{[1]}} \otimes \cdots \otimes \varphi_{T^{[g]}V^{[g]}}) \\ &= \varphi_{U^{[1]}S^{[1]}} \varphi_{T^{[1]}V^{[1]}} \otimes \cdots \otimes \varphi_{U^{[g]}S^{[g]}} \varphi_{T^{[g]}V^{[g]}} \\ &\equiv \langle \varphi_{S^{[1]}}, \varphi_{T^{[1]}} \rangle \varphi_{U^{[1]}V^{[1]}} \otimes \cdots \otimes \langle \varphi_{S^{[g]}}, \varphi_{T^{[g]}} \rangle \varphi_{U^{[g]}V^{[g]}} \\ &\quad \text{mod } \mathcal{S}(A_{n_1})^{\vee \lambda^{[1]}} \otimes \cdots \otimes \mathcal{S}(A_{n_g})^{\vee \lambda^{[g]}} \\ &= \langle \varphi_{S^{[1]}}, \varphi_{T^{[1]}} \rangle \cdots \langle \varphi_{S^{[g]}}, \varphi_{T^{[g]}} \rangle \varphi_{U^{[1]}V^{[1]}} \otimes \cdots \otimes \varphi_{U^{[g]}V^{[g]}} \\ &= \langle \varphi_{S^{[1]}}, \varphi_{T^{[1]}} \rangle \cdots \langle \varphi_{S^{[g]}}, \varphi_{T^{[g]}} \rangle \overline{\varphi}_{UV}. \end{aligned}$$

Since $\mathcal{S}(A_{n_1})^{\vee \lambda^{[1]}} \otimes \cdots \otimes \mathcal{S}(A_{n_g})^{\vee \lambda^{[g]}} \subset (\overline{\mathcal{S}}^{\mathbf{p}})^{\vee \lambda}$, we see that

$$\langle \overline{\varphi}_S, \overline{\varphi}_T \rangle_{\mathbf{p}} \overline{\varphi}_{UV} \equiv \overline{\varphi}_{US} \overline{\varphi}_{TV} \equiv \langle \varphi_{S^{[1]}}, \varphi_{T^{[1]}} \rangle \cdots \langle \varphi_{S^{[g]}}, \varphi_{T^{[g]}} \rangle \overline{\varphi}_{UV} \quad \text{mod } (\overline{\mathcal{S}}^{\mathbf{p}})^{\vee \lambda}.$$

The lemma is proved. \square

Remark 1.10. For the isomorphism in (1.8.3), we do not need to assume that R is a field. But for (1.8.4) and (1.8.5), we need that R is a field.

Theorem 1.11. [SW, Theorem 4.17] *Assume that R is a field. For $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have the following.*

$$d_{\lambda\mu} = \overline{d}_{\lambda\mu} = \prod_{k=1}^g d_{\lambda^{[k]}\mu^{[k]}},$$

where $d_{\lambda^{[k]}\mu^{[k]}} = [W^{\lambda^{[k]}} : L^{\mu^{[k]}}]$.

2. DECOMPOSITION NUMBERS WITH JANTZEN FILTRATION

2.1. In the rest of this paper, we assume that R is a discrete valuation ring. Let \wp be a unique maximal ideal of R and $F = R/\wp$ be the residue field. Fix $\widehat{q}, \widehat{Q}_1, \dots, \widehat{Q}_r$ in R and let $q = \widehat{q} + \wp, Q_1 = \widehat{Q}_1 + \wp, \dots, Q_r = \widehat{Q}_r + \wp$ be their canonical images in F . Moreover let K be the quotient field of R . Then (K, R, F) is a modular system with parameters. Let $\mathcal{S}_R = \mathcal{S}_R(\Lambda)$ be the cyclotomic \widehat{q} -Schur algebra over R with parameters $\widehat{q}, \widehat{Q}_1, \dots, \widehat{Q}_r$ and $\mathcal{S} = \mathcal{S}(\Lambda)$ be the cyclotomic q -Schur algebra over F with parameters q, Q_1, \dots, Q_r . Then $\mathcal{S} = (\mathcal{S}_R + \wp \mathcal{S}_R) / \wp \mathcal{S}_R$.

We consider the subalgebra $\mathcal{S}_R^{\mathbf{p}}$ (resp. $\mathcal{S}^{\mathbf{p}}$) of \mathcal{S}_R (resp. \mathcal{S}) and its quotient $\overline{\mathcal{S}}_R^{\mathbf{p}}$ (resp. $\overline{\mathcal{S}}^{\mathbf{p}}$) as in the previous section with the notation there. Note that the subscript R is used to indicate the objects related to R .

For $\lambda \in \Lambda^+$, let W_R^λ be the Weyl module of \mathcal{S}_R . For $i \in \mathbb{Z}_{\geq 0}$, we set

$$W_R^\lambda(i) = \{x \in W_R^\lambda \mid \langle x, y \rangle \in \wp^i \text{ for any } y \in W_R^\lambda\}$$

and define

$$W^\lambda(i) = (W_R^\lambda(i) + \wp W_R^\lambda) / \wp W_R^\lambda.$$

Then $W^\lambda = W^\lambda(0)$ is the Weyl module of \mathcal{S} , and we have the Jantzen filtration of W^λ ,

$$W^\lambda = W^\lambda(0) \supset W^\lambda(1) \supset W^\lambda(2) \supset \dots$$

Similarly, by using the bilinear form $\langle \cdot, \cdot \rangle_{\mathbf{p}}$ on \overline{Z}^λ , one can define the Jantzen filtration of $\overline{\mathcal{S}}^{\mathbf{p}}$ -module \overline{Z}^λ

$$\overline{Z}^\lambda = \overline{Z}^\lambda(0) \supset \overline{Z}^\lambda(1) \supset \overline{Z}^\lambda(2) \supset \dots$$

Moreover for the Weyl module $Z_R^{(\lambda,0)}$ of $\mathcal{S}_R^{\mathbf{p}}$, we set

$$Z_R^{(\lambda,0)}(i) = \{x \in Z_R^{(\lambda,0)} \mid \beta_\lambda(y, x) \in \wp^i \text{ for any } y \in \diamond Z_R^{(\lambda,0)}\}$$

and define

$$Z^{(\lambda,0)}(i) = (Z_R^{(\lambda,0)}(i) + \wp Z_R^{(\lambda,0)}) / \wp Z_R^{(\lambda,0)}.$$

Then we have the Jantzen filtration of $Z^{(\lambda,0)}$

$$Z^{(\lambda,0)} = Z^{(\lambda,0)}(0) \supset Z^{(\lambda,0)}(1) \supset Z^{(\lambda,0)}(2) \supset \dots$$

Since W^λ is a finite dimensional F -vector space, one can find a positive integer k such that $W^\lambda(k') = W^\lambda(k)$ for any $k' > k$. We choose a minimal k in such numbers and set $W^\lambda(k+1) = 0$. Then the Jantzen filtration of W^λ becomes a finite sequence. Similarly, Jantzen filtrations of $Z^{(\lambda,0)}$ and \overline{Z}^λ also become finite sequences.

We can easily see that $W^\lambda(i)$ (resp. $Z^{(\lambda,0)}(i)$, $\overline{Z}^\lambda(i)$) is a \mathcal{S} -submodule of W^λ (resp. $\mathcal{S}^{\mathbf{p}}$ -submodule of $Z^{(\lambda,0)}$, $\overline{\mathcal{S}}^{\mathbf{p}}$ -submodule of \overline{Z}^λ) by associativity of the bilinear form (1.3.1) (resp. (1.4.1), (1.5.1)).

2.2. Take $\lambda, \mu \in \Lambda^+$, and $W^\lambda = W^\lambda(0) \supset W^\lambda(1) \supset \dots$ be the Jantzen filtration of W^λ . Let $[W^\lambda(i)/W^\lambda(i+1) : L^\mu]$ be the composition multiplicity of L^μ in

$W^\lambda(i)/W^\lambda(i+1)$. Let v be an indeterminate. We define a polynomial $d_{\lambda\mu}(v)$ by

$$d_{\lambda\mu}(v) = \sum_{i \geq 0} [W^\lambda(i)/W^\lambda(i+1) : L^\mu] \cdot v^i.$$

Similarly we define, for $Z^{(\lambda,0)}$ and \overline{Z}^λ ,

$$d_{\lambda\mu}^{(\lambda,0)}(v) = \sum_{i \geq 0} [Z^{(\lambda,0)}(i)/Z^{(\lambda,0)}(i+1) : L^{(\mu,0)}] \cdot v^i,$$

$$\overline{d}_{\lambda\mu}(v) = \sum_{i \geq 0} [\overline{Z}^\lambda(i)/\overline{Z}^\lambda(i+1) : \overline{L}^\mu] \cdot v^i.$$

Thus $d_{\lambda\mu}(v)$, $d_{\lambda\mu}^{(\lambda,0)}(v)$ and $\overline{d}_{\lambda\mu}(v)$ are polynomials whose coefficients are non-negative integers. Note that since the Jantzen filtration of W^λ , etc. are finite sequences, these summations are finite sums. We call $d_{\lambda\mu}(v)$ (resp. $d_{\lambda\mu}^{(\lambda,0)}(v)$, $\overline{d}_{\lambda\mu}(v)$) decomposition number with Jantzen filtration of \mathcal{S} (resp. $\mathcal{S}^{\mathbf{p}}$, $\overline{\mathcal{S}}^{\mathbf{p}}$). We also call them v -decomposition numbers as they coincide at $v = 1$ with decomposition numbers given in 1.6.

We have the following relation between $d_{\lambda\mu}^{(\lambda,0)}(v)$ and $\overline{d}_{\lambda\mu}(v)$.

Proposition 2.3. *For $\lambda, \mu \in \Lambda$, we have*

- (i) *If $\alpha_{\mathbf{p}}(\lambda) \neq \alpha_{\mathbf{p}}(\mu)$, then $\overline{d}_{\lambda\mu}(v) = d_{\lambda\mu}^{(\lambda,0)}(v) = 0$.*
- (ii) *$[\overline{Z}^\lambda(i)/\overline{Z}^\lambda(i+1) : \overline{L}^\mu] = [Z^{(\lambda,0)}(i)/Z^{(\lambda,0)}(i+1) : L^{(\mu,0)}]$ for any $i \geq 0$.*
Hence we have $\overline{d}_{\lambda\mu}(v) = d_{\lambda\mu}^{(\lambda,0)}(v)$.

Proof. (i) is clear since $d_{\lambda\mu}^{(\lambda,0)} = \overline{d}_{\lambda\mu} = 0$ by [SW, Proposition 3.3].

Recall that $\overline{Z}^\lambda \cong Z^{(\lambda,0)}$ and $\overline{L}^\mu \cong L^{(\mu,0)}$ as $\mathcal{S}^{\mathbf{p}}$ -modules by [SW, Lemma 3.2]. By definition, we have $\beta_\lambda(\varphi_S^{(\lambda,0)}, \varphi_T^{(\lambda,0)}) = \langle \overline{\varphi}_T, \overline{\varphi}_S \rangle_{\mathbf{p}}$ for any $S, T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)$. Then under the isomorphism $\overline{Z}^\lambda \cong Z^{(\lambda,0)}$, the Jantzen filtration of \overline{Z}^λ coincides with that of $Z^{(\lambda,0)}$. So (ii) is proved. \square

2.4. Next, we consider the relation between $d_{\lambda\mu}^{(\lambda,0)}(v)$ and $d_{\lambda\mu}(v)$. In order to see this we prepare two lemmas. Recall that there exists an injective $\mathcal{S}^{\mathbf{p}}$ -homomorphism $f_\lambda : Z^{(\lambda,0)} \hookrightarrow W^\lambda$ such that $f_\lambda(\varphi_T^{(\lambda,0)}) = \varphi_T$ for $T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)$ by [SW, Lemma 3.5] and that $Z^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S} \cong W^\lambda$ as \mathcal{S} -module by [SW, Proposition 3.6]. Let $\iota_i : Z^{(\lambda,0)}(i) \hookrightarrow Z^{(\lambda,0)}$ be an inclusion map. Then $(\iota_i \otimes \text{id}_{\mathcal{S}})(Z^{(\lambda,0)}(i) \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S})$ is the \mathcal{S} -submodule of $Z^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}$. Similar results hold also for R . We have the following.

Lemma 2.5. *Let $\lambda \in \Lambda^+$. For any $i \geq 0$, we have*

$$f_\lambda^{-1}(W^\lambda(i)) = Z^{(\lambda,0)}(i)$$

Proof. By definition, we see that $\beta(\varphi_T^{(\lambda,0)}, \varphi_S^{(\lambda,0)}) = \langle \varphi_S, \varphi_T \rangle$ for any $S, T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)$, and that $\langle \varphi_S, \varphi_T \rangle = 0$ if $S \in \mathcal{T}_0^{\mathbf{p}}(\lambda)$, $T \in \mathcal{T}_0(\lambda) \setminus \mathcal{T}_0^{\mathbf{p}}(\lambda)$. Then for $x \in Z_R^{(\lambda,0)}$, we

have

$$\begin{aligned} x \in Z_R^{(\lambda,0)}(i) &\Leftrightarrow \beta_\lambda(\varphi_T^{(\lambda,0)}, x) \in \wp^i \quad \text{for any } T \in \mathcal{T}_0^{\mathbf{p}}(\lambda) \\ &\Leftrightarrow \langle f_\lambda(x), \varphi_T \rangle \in \wp^i \quad \text{for any } T \in \mathcal{T}_0(\lambda) \\ &\Leftrightarrow f_\lambda(x) \in W_R^\lambda(i) \end{aligned}$$

By taking the quotient, we obtain the lemma. \square

Lemma 2.6. *Let $\lambda \in \Lambda^+$. For any $i \geq 0$, we have*

$$(\iota_i \otimes \text{id}_{\mathcal{S}})(Z^{(\lambda,0)}(i) \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}) \subset W^\lambda(i)$$

under the isomorphism $Z^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S} \cong W^\lambda$.

Proof. Recall that any element of $Z_R^{(\lambda,0)}$ can be written in the form $\varphi_{T^\lambda}^{(\lambda,0)} \cdot \psi$ with $\psi \in \mathcal{S}_R^{\mathbf{p}}$. Moreover it follows from [SW, Proposition 3.6] that, under the isomorphism $g_\lambda : Z_R^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}_R \xrightarrow{\sim} W_R^\lambda$, we have $g_\lambda(\varphi_{T^\lambda}^{(\lambda,0)} \cdot \psi \otimes \varphi) = \varphi_{T^\lambda} \cdot \psi \varphi$ for $\psi \in \mathcal{S}_R^{\mathbf{p}}$, $\varphi \in \mathcal{S}_R$. This is true also for $Z^{(\lambda,0)}$ and W^λ . Thus in order to show the lemma, it is enough to prove the following.

(2.6.1) Suppose that $\varphi_{T^\lambda}^{(\lambda,0)} \psi \in Z^{(\lambda,0)}(i)$ for $\psi \in \mathcal{S}^{\mathbf{p}}$. Then we have $\varphi_{T^\lambda} \psi \varphi \in W^\lambda(i)$ for any $\varphi \in \mathcal{S}$.

Now take $\varphi_{T^\lambda}^{(\lambda,0)} \psi \in Z_R^{(\lambda,0)}$. If $\varphi_{T^\lambda}^{(\lambda,0)} \cdot \psi \in Z_R^{(\lambda,0)}(i)$, then $\beta_\lambda(x, \varphi_{T^\lambda}^{(\lambda,0)} \psi) \in \wp^i$ for any $x \in \diamond Z_R^{(\lambda,0)}$. This implies that $\langle \varphi_{T^\lambda} \psi, y \rangle \in \wp^i$ for any $y \in W_R^\lambda$ by a similar argument as the proof of Lemma 2.5.

Since $\langle \varphi_{T^\lambda} \psi \varphi, y \rangle = \langle \varphi_{T^\lambda} \psi, y \varphi^* \rangle$ for any $y \in W_R^\lambda$ and any $\varphi \in \mathcal{S}_R$, we see that $\varphi_{T^\lambda}^{(\lambda,0)} \psi \in Z_R^{(\lambda,0)}(i)$ implies that $\varphi_{T^\lambda} \psi \varphi \in W_R^\lambda(i)$. By taking the quotient, we obtain (2.6.1). Thus the lemma is proved. \square

These two lemmas imply the following proposition about the relation between $d_{\lambda\mu}^{(\lambda,0)}(v)$ and $d_{\lambda\mu}(v)$.

Proposition 2.7. *Let $\lambda, \mu \in \Lambda^+$ be such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$. Then for any $i \geq 0$, we have*

$$[Z^{(\lambda,0)}(i)/Z^{(\lambda,0)}(i+1) : L^{(\mu,0)}] = [W^\lambda(i)/W^\lambda(i+1) : L^\mu].$$

Hence we have $d_{\lambda\mu}^{(\lambda,0)}(v) = d_{\lambda\mu}(v)$ if $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$.

Proof. Fix $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, and an integer $i \geq 0$. Thanks to Lemma 2.5, we have the following result by similar arguments as in the proof of [SW, Proposition 3.12].

$$(2.7.1) \quad [Z^{(\lambda,0)}(i)/Z^{(\lambda,0)}(i+1) : L^{(\mu,0)}] \geq [W^\lambda(i)/W^\lambda(i+1) : L^\mu].$$

Conversly, thanks to Lemma 2.6, we have the following result by similar arguments as in the proof of [SW, Proposition 3.11].

$$(2.7.2) \quad [Z^{(\lambda,0)}(i) : L^{(\mu,0)}] \leq [W^\lambda(i) : L^\mu].$$

We remark that this does not imply dilectry

$$[Z^{(\lambda,0)}(i)/Z^{(\lambda,0)}(i+1) : L^{(\mu,0)}] \leq [W^\lambda(i)/W^\lambda(i+1) : L^\mu]$$

since we can not see whether $(\iota_{i+1} \otimes \text{id}_{\mathcal{S}})(Z^{(\lambda,0)}(i+1) \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}) = W^\lambda(i+1)$ or not. Instead, we argue as follows. Let

$$W^\lambda = W^\lambda(0) \supset W^\lambda(1) \supset \cdots \supset W^\lambda(k) \supsetneq W^\lambda(k+1) = 0$$

$$Z^{(\lambda,0)} = Z^{(\lambda,0)}(0) \supset Z^{(\lambda,0)}(1) \supset \cdots \supset Z^{(\lambda,0)}(l) \supsetneq Z^{(\lambda,0)}(l+1) = 0$$

be the Jantzen filtrations of W^λ and $Z^{(\lambda,0)}$ respectively. Then we have

$$(\iota_{k+1} \otimes \text{id}_{\mathcal{S}})(Z^{(\lambda,0)}(k+1) \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}) \subset W^\lambda(k+1) = 0$$

by Lemma 2.6. This implies that $Z^{(\lambda,0)}(k+1) = 0$ since $(\iota \otimes \text{id}_{\mathcal{S}})(M \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}) \neq 0$ for any non-zero submodule M of $Z^{(\lambda,0)}$ and the inclusion map $\iota : M \hookrightarrow Z^{(\lambda,0)}$ by [SW, Lemma 3.8 (ii)]. So we have $l \leq k$.

Now, if L^μ is a composition factor of $W^\lambda(i)/W^\lambda(i+1)$, then we have

$$1 \leq [W^\lambda(i)/W^\lambda(i+1) : L^\mu] \leq [Z^{(\lambda,0)}(i)/Z^{(\lambda,0)}(i+1) : L^{(\mu,0)}]$$

by (2.7.1). Hence, we have $Z^{(\lambda,0)}(i) \neq 0$. This implies that $i \leq l$ if L^μ is a composition factor of $W^\lambda(i)$. In particular, $[W^\lambda(l+1) : L^\mu] = 0$. Thus we have

$$[Z^{(\lambda,0)}(l)/Z^{(\lambda,0)}(l+1) : L^{(\mu,0)}] = [Z^{(\lambda,0)}(l) : L^{(\mu,0)}]$$

$$[W^\lambda(l)/W^\lambda(l+1) : L^\mu] = [W^\lambda(l) : L^\mu].$$

Combining these equalities with (2.7.1) and (2.7.2), we have

$$[Z^{(\lambda,0)}(l)/Z^{(\lambda,0)}(l+1) : L^{(\mu,0)}] = [W^\lambda(l)/W^\lambda(l+1) : L^\mu],$$

and so

$$(2.7.3) \quad [Z^{(\lambda,0)}(l) : L^{(\mu,0)}] = [W^\lambda(l) : L^\mu].$$

Next we consider the case where $i = l - 1$. Note that

$$[Z^{(\lambda,0)}(l-1)/Z^{(\lambda,0)}(l) : L^{(\mu,0)}] = [Z^{(\lambda,0)}(l-1) : L^{(\mu,0)}] - [Z^{(\lambda,0)}(l) : L^{(\mu,0)}],$$

$$[W^\lambda(l-1)/W^\lambda(l) : L^\mu] = [W^\lambda(l-1) : L^\mu] - [W^\lambda(l) : L^\mu].$$

Combined with (2.7.1), (2.7.2) and (2.7.3), we have

$$[Z^{(\lambda,0)}(l-1)/Z^{(\lambda,0)}(l) : L^{(\mu,0)}] = [W^\lambda(l-1)/W^\lambda(l) : L^\mu]$$

and so $[Z^{(\lambda,0)}(l-1) : L^{(\mu,0)}] = [W^\lambda(l-1) : L^\mu]$. Therefore by backward induction on l , we obtain the proposition. \square

Combining Proposition 2.3 and Proposition 2.7, we have the following theorem.

Theorem 2.8. *For any $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have*

$$\bar{d}_{\lambda\mu}(v) = d_{\lambda\mu}^{(\lambda,0)}(v) = d_{\lambda\mu}(v).$$

If we specialize $v = 1$, the theorem reduces to Theorem 1.7.

2.9. For later use, we shall consider the basis of $W_R^\lambda(i)$, following the computation in the proof of [M2, Lemma 5.30]. Let π be the generator of \wp , namely $\wp = (\pi)$, and ν_\wp be the valuation map on R . Let $G^\lambda = (\langle \varphi_S, \varphi_T \rangle)_{S,T \in \mathcal{T}_0(\lambda)}$ be the Gram matrix of W_R^λ . Since R is a PID, there exist $P, Q \in \text{GL}_N(R)$ (where $N = |\mathcal{T}_0(\lambda)|$) such that $PG^\lambda Q = \text{diag}(d_{S_1}, d_{S_2}, \dots, d_{S_N})$, where $d_{S_k} \in R$ and $\{S_1, \dots, S_N\} = \mathcal{T}_0(\lambda)$. Let $P = (p_{ST})_{S,T \in \mathcal{T}_0(\lambda)}$, $Q = (q_{ST})_{S,T \in \mathcal{T}_0(\lambda)}$ and we define, for $S, T \in \mathcal{T}_0(\lambda)$,

$$f_S = \sum_{S' \in \mathcal{T}_0(\lambda)} p_{SS'} \varphi_{S'}, \quad g_T = \sum_{T' \in \mathcal{T}_0(\lambda)} q_{T'T} \varphi_{T'}.$$

Since both P and Q are regular matrices, $\{f_S \mid S \in \mathcal{T}_0(\lambda)\}$ and $\{g_T \mid T \in \mathcal{T}_0(\lambda)\}$ are basis of W_R^λ respectively. Moreover we have $\text{diag}(d_{S_1}, \dots, d_{S_N}) = PG^\lambda Q = (\langle f_S, g_T \rangle)_{S,T \in \mathcal{T}_0(\lambda)}$ by definition. Thus we have

$$(2.9.1) \quad \langle f_S, g_T \rangle = \delta_{ST} d_S \quad (S, T \in \mathcal{T}_0(\lambda))$$

where $\delta_{ST} = 1$ if $S = T$ and $\delta_{ST} = 0$ otherwise. For $x = \sum_{S \in \mathcal{T}_0(\lambda)} r_S f_S \in W_R^\lambda$ ($r_S \in R$), we have

$$\begin{aligned} x \in W_R^\lambda(i) &\Leftrightarrow \langle x, g_T \rangle \in \wp^i \quad \text{for any } T \in \mathcal{T}_0(\lambda) \\ &\Leftrightarrow r_T d_T \in \wp^i \quad \text{for any } T \in \mathcal{T}_0(\lambda) \quad (\text{by (2.9.1)}) \\ &\Leftrightarrow \nu_\wp(r_T d_T) = \nu_\wp(r_T) + \nu_\wp(d_T) \geq i \quad \text{for any } T \in \mathcal{T}_0(\lambda). \end{aligned}$$

It follows from this that $W_R^\lambda(i)$ is a free R -module with basis

$$(2.9.2) \quad \{f_T \mid T \in \mathcal{T}_0(\lambda), \nu_\wp(d_T) \geq i\} \cup \{\pi^{i-\nu_\wp(d_T)} f_T \mid T \in \mathcal{T}_0(\lambda), \nu_\wp(d_T) < i\}.$$

2.10. We consider the Jantzen filtration of $W^{\lambda[k]}$ ($1 \leq k \leq g$) as in the case of W^λ and use the notation similar to the case of W^λ . Since we see that $W_R^{\lambda[k]}(i_k)$ ($i_k \geq 0$) is a free R -module (see 2.9.), $W_R^{\lambda[1]}(i_1) \otimes \dots \otimes W_R^{\lambda[g]}(i_g)$ ($(i_1, \dots, i_g) \in Z_{\geq 0}^g$) becomes the submodule of $W_R^{\lambda[1]} \otimes \dots \otimes W_R^{\lambda[g]}$.

For $1 \leq k \leq g$, let $\{f_{S^{[k]}} \mid S^{[k]} \in \mathcal{T}_0(\lambda^{[k]})\}$ and $\{g_{T^{[k]}} \mid T^{[k]} \in \mathcal{T}_0(\lambda^{[k]})\}$ be the bases of $W_R^{\lambda^{[k]}}$ as of W_R^λ in 2.9. For $S, T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)$, we define $\bar{f}_S := f_{S^{[1]}} \otimes \dots \otimes f_{S^{[g]}}$ and $\bar{g}_T := g_{T^{[1]}} \otimes \dots \otimes g_{T^{[g]}}$. Then $\{\bar{f}_S \mid S \in \mathcal{T}_0^{\mathbf{p}}(\lambda)\}$ and $\{\bar{g}_T \mid T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)\}$ turn

out to be the bases of \overline{Z}_R^λ . By Lemma 1.9 and (2.9.1), we have

$$(2.10.1) \quad \langle \overline{f}_S, \overline{g}_T \rangle_{\mathbf{P}} = \delta_{ST} d_{T[1]} \cdots d_{T[g]} \quad \text{for } S, T \in \mathcal{T}_0^{\mathbf{P}}(\lambda).$$

We set $d_T = d_{T[1]} \cdots d_{T[g]}$. Then we have the following result by a similar argument as in 2.9. $\overline{Z}_R^\lambda(i)$ is a free R -module with basis

$$(2.10.2) \quad \{ \overline{f}_T \mid T \in \mathcal{T}_0^{\mathbf{P}}(\lambda), \nu_\varphi(d_T) \geq i \} \cup \{ \pi^{i-\nu_\varphi(d_T)} \overline{f}_T \mid T \in \mathcal{T}_0^{\mathbf{P}}(\lambda), \nu_\varphi(d_T) < i \}.$$

Recall that $\Delta_{i,g}$ is the set of $(i_1, \dots, i_g) \in \mathbb{Z}_{\geq 0}^g$ such that $i_1 + \cdots + i_g = i$. Then we have the following proposition.

Proposition 2.11. *Let $\lambda \in \Lambda^+$ and $i \geq 0$. Under the isomorphism $\overline{Z}_R^\lambda \cong W_R^{\lambda[1]} \otimes \cdots \otimes W_R^{\lambda[g]}$, we have*

$$(2.11.1) \quad \overline{Z}_R^\lambda(i) = \sum_{(i_1, \dots, i_g) \in \Delta_{i,g}} W_R^{\lambda[1]}(i_1) \otimes \cdots \otimes W_R^{\lambda[g]}(i_g)$$

Proof. First we show that the right hand side is contained in the left hand side. Take $x = x^{[1]} \otimes \cdots \otimes x^{[g]} \in W_R^{\lambda[1]}(i_1) \otimes \cdots \otimes W_R^{\lambda[g]}(i_g)$ such that $i_1 + \cdots + i_g = i$. By Lemma 1.9, we have

$$\begin{aligned} \langle x, \overline{\varphi}_T \rangle_{\mathbf{P}} &= \langle x^{[1]}, \varphi_{T[1]} \rangle \cdots \langle x^{[g]}, \varphi_{T[g]} \rangle \\ &\in \wp^{i_1} \cdots \wp^{i_g} = \wp^i \quad \text{for any } T \in \mathcal{T}_0^{\mathbf{P}}(\lambda). \end{aligned}$$

Thus we have $x \in \overline{Z}_R^\lambda(i)$.

Then in order to show the equality, we have only to show that the basis element of $\overline{Z}_R^\lambda(i)$ is contained in the right hand side of (2.11.1). First, we consider \overline{f}_T such that $\nu_\varphi(d_T) \geq i$. Since $\nu_\varphi(d_T) = \nu_\varphi(d_{T[1]}) + \cdots + \nu_\varphi(d_{T[g]})$, one can find $(i_1, \dots, i_g) \in \mathbb{Z}_{\geq 0}^g$ such that $i_1 + \cdots + i_g = i$ and that $\nu_\varphi(d_{T[k]}) \geq i_k$ for $k = 1, \dots, g$. Then

$$\overline{f}_T = f_{T[1]} \otimes \cdots \otimes f_{T[g]} \in W_R^{\lambda[1]}(i_1) \otimes \cdots \otimes W_R^{\lambda[g]}(i_g),$$

and so \overline{f}_T is contained in the right hand side of (2.11.1).

Next we consider \overline{f}_T such that $\nu_\varphi(d_T) < i$. Then one can find (i_1, \dots, i_g) such that $i_1 + \cdots + i_g = i$ and that $\nu_\varphi(d_{T[k]}) \leq i_k$ for $k = 1, \dots, g$. Therefore $\pi^{i-\nu_\varphi(d_T)} \overline{f}_T = (\pi^{i_1-\nu_\varphi(d_{T[1]})} f_{T[1]}) \otimes \cdots \otimes (\pi^{i_g-\nu_\varphi(d_{T[g]})} f_{T[g]})$ is also contained in the right hand side of (2.11.1). The proposition is proved. \square

We have the following corollary.

Corollary 2.12. *For $\lambda \in \Lambda^+$ and $i \geq 0$, under the isomorphism $\overline{Z}^\lambda \cong W^{\lambda[1]} \otimes \cdots \otimes W^{\lambda[g]}$, we have*

$$\overline{Z}^\lambda(i) = \sum_{(i_1, \dots, i_g) \in \Delta_{i,g}} W^{\lambda[1]}(i_1) \otimes \cdots \otimes W^{\lambda[g]}(i_g)$$

Proof. By definition, we have

$$\overline{Z}^\lambda(i) = (\overline{Z}_R^\lambda(i) + \wp \overline{Z}_R^\lambda) / \wp \overline{Z}_R^\lambda \cong \overline{Z}_R^\lambda(i) / (\overline{Z}_R^\lambda(i) \cap \wp \overline{Z}_R^\lambda)$$

and

$$\begin{aligned} W^{\lambda^{[1]}}(i_1) \otimes \cdots \otimes W^{\lambda^{[g]}}(i_g) \\ &= (W_R^{\lambda^{[1]}}(i_1) + \wp W_R^{\lambda^{[1]}}) / \wp W_R^{\lambda^{[1]}} \otimes \cdots \otimes (W_R^{\lambda^{[g]}}(i_g) + \wp W_R^{\lambda^{[g]}}) / \wp W_R^{\lambda^{[g]}} \\ &\cong W_R^{\lambda^{[1]}}(i_1) / (W_R^{\lambda^{[1]}}(i_1) \cap \wp W_R^{\lambda^{[1]}}) \otimes \cdots \otimes W_R^{\lambda^{[g]}}(i_g) / (W_R^{\lambda^{[g]}}(i_g) \cap \wp W_R^{\lambda^{[g]}}). \end{aligned}$$

By Proposition 2.11, we have a surjective map

$$\Phi : \overline{Z}_R^\lambda(i) \rightarrow \sum_{(i_1, \dots, i_g) \in \Delta_{i,g}} W^{\lambda^{[1]}}(i_1) \otimes \cdots \otimes W^{\lambda^{[g]}}(i_g).$$

We claim that $\text{Ker } \Phi = \overline{Z}_R^\lambda(i) \cap \wp \overline{Z}_R^\lambda$. Then the claim implies the corollary. So we shall show the claim. By definition, it is clear that $\text{Ker } \Phi$ is contained in $\overline{Z}_R^\lambda(i) \cap \wp \overline{Z}_R^\lambda$. Take $x = \sum_{T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} r_T \overline{f}_T \in \overline{Z}_R^\lambda(i) \cap \wp \overline{Z}_R^\lambda$. Then $r_T \in \wp$ for any $T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)$. If $\nu_\wp(d_T) \geq i$, then $\overline{f}_T = f_{T^{[1]}} \otimes \cdots \otimes f_{T^{[g]}}$ is contained in $W_R^{\lambda^{[1]}}(i_1) \otimes \cdots \otimes W_R^{\lambda^{[g]}}(i_g)$ for some $(i_1, \dots, i_g) \in \Delta_{i,g}$ by the proof of Proposition 2.11. So we have $r_T \overline{f}_T \in \text{Ker } \Phi$ for $T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)$ such that $\nu_\wp(d_T) \geq i$. If $\nu_\wp(d_T) < i$, then $r_T \overline{f}_T = r'_T \pi^{i-\nu_\wp(d_T)} \overline{f}_T$ for some $r'_T \in R$ since $x \in \overline{Z}_R^\lambda(i)$. By the proof of Proposition 2.11, for some $(i_1, \dots, i_g) \in \Delta_{i,g}$, $\pi^{i-\nu_\wp(d_T)} \overline{f}_T = (\pi^{i_1-\nu_\wp(d_{T^{[1]}})} f_{T^{[1]}}) \otimes \cdots \otimes (\pi^{i_g-\nu_\wp(d_{T^{[g]}})} f_{T^{[g]}})$ is contained in $W_R^{\lambda^{[1]}}(i_1) \otimes \cdots \otimes W_R^{\lambda^{[g]}}(i_g)$. Moreover one can find at least one k such that $\nu_\wp(d_{T^{[k]}}) < i_k$. Then the image of $\pi^{i_k-\nu_\wp(d_{T^{[k]}})} f_{T^{[k]}}$ in $W_R^{\lambda^{[k]}}(i_k) / (W_R^{\lambda^{[k]}}(i_k) \cap \wp W^{\lambda^{[k]}})$ is zero. Hence for $T \in \mathcal{T}_0^{\mathbf{p}}$ such that $\nu_\wp(d_T) < i$, $r_T \overline{f}_T$ is also contained in $\text{Ker } \Phi$. Now the claim is proved, and the corollary follows. \square

By using the corollary, we show the following lemma.

Lemma 2.13. *Let $\lambda, \mu \in \Lambda^+$ be such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$. For any $i \geq 0$, we have*

$$[\overline{Z}^\lambda(i) / \overline{Z}^\lambda(i+1) : \overline{L}^\mu] = \sum_{(i_1, \dots, i_g) \in \Delta_{i,g}} \prod_{k=1}^g [W^{\lambda^{[k]}}(i_k) / W^{\lambda^{[k]}}(i_{k+1}) : L^{\mu^{[k]}}]$$

Proof. By Corollary 2.12, we have

$$\begin{aligned} (2.13.1) \quad \overline{Z}^\lambda(i) / \overline{Z}^\lambda(i+1) &= \left(\sum_{(i_1, \dots, i_g) \in \Delta_{i,g}} W^{\lambda^{[1]}}(i_1) \otimes \cdots \otimes W^{\lambda^{[g]}}(i_g) \right) / \overline{Z}^\lambda(i+1) \\ &= \sum_{(i_1, \dots, i_g) \in \Delta_{i,g}} (W^{\lambda^{[1]}}(i_1) \otimes \cdots \otimes W^{\lambda^{[g]}}(i_g)) / \left(\overline{Z}^\lambda(i+1) \cap (W^{\lambda^{[1]}}(i_1) \otimes \cdots \otimes W^{\lambda^{[g]}}(i_g)) \right). \end{aligned}$$

If $(i_1, \dots, i_g) \neq (j_1, \dots, j_g)$ such that $i_1 + \dots + i_g = j_1 + \dots + j_g = i$, then

$$(W^{\lambda^{[1]}}(i_1) \otimes \dots \otimes W^{\lambda^{[g]}}(i_g)) \cap (W^{\lambda^{[1]}}(j_1) \otimes \dots \otimes W^{\lambda^{[g]}}(j_g)) \subset W^{\lambda^{[1]}}(k_1) \otimes \dots \otimes W^{\lambda^{[g]}}(k_g),$$

where $k_l = \max\{i_l, j_l\}$. Since $(i_1, \dots, i_g) \neq (j_1, \dots, j_g)$, $k_1 + \dots + k_g \geq i + 1$. Hence we have

$$(W^{\lambda^{[1]}}(i_1) \otimes \dots \otimes W^{\lambda^{[g]}}(i_g)) \cap (W^{\lambda^{[1]}}(j_1) \otimes \dots \otimes W^{\lambda^{[g]}}(j_g)) \subset \overline{Z}^\lambda(i + 1).$$

It follows from this, we see that the sum in (2.13.1) is a direct sum.

For $(i_1, \dots, i_g) \in \Delta_{i,g}$, we consider a surjective $\overline{\mathcal{S}}^{\mathbf{p}}$ -homomorphism

$$\Psi : W^{\lambda^{[1]}}(i_1) \otimes \dots \otimes W^{\lambda^{[g]}}(i_g) \rightarrow W^{\lambda^{[1]}}(i_1)/W^{\lambda^{[1]}}(i_1 + 1) \otimes \dots \otimes W^{\lambda^{[g]}}(i_g)/W^{\lambda^{[g]}}(i_g + 1)$$

Then we have $\text{Ker } \Psi = \overline{Z}^\lambda(i + 1) \cap (W^{\lambda^{[1]}}(i_1) \otimes \dots \otimes W^{\lambda^{[g]}}(i_g))$ under the setting in Corollary 2.12. By noting that (2.13.1) is a direct sum, we have

$$\overline{Z}^\lambda(i)/\overline{Z}^\lambda(i + 1) \cong \bigoplus_{(i_1, \dots, i_g) \in \Delta_{i,g}} \left(W^{\lambda^{[1]}}(i_1)/W^{\lambda^{[1]}}(i_1 + 1) \otimes \dots \otimes W^{\lambda^{[g]}}(i_g)/W^{\lambda^{[g]}}(i_g + 1) \right)$$

Since $\overline{L}^\mu \cong L^{\mu^{[1]}} \otimes \dots \otimes L^{\mu^{[g]}}$, we have

$$\begin{aligned} \left[\overline{Z}^\lambda(i)/\overline{Z}^\lambda(i + 1) : \overline{L}^\mu \right] &= \sum_{(i_1, \dots, i_g) \in \Delta_{i,g}} \left[W^{\lambda^{[1]}}(i_1)/W^{\lambda^{[1]}}(i_1 + 1) \otimes \dots \otimes W^{\lambda^{[g]}}(i_g)/W^{\lambda^{[g]}}(i_g + 1) : \overline{L}^\mu \right] \\ &= \sum_{(i_1, \dots, i_g) \in \Delta_{i,g}} \prod_{k=1}^g \left[W^{\lambda^{[k]}}(i_k)/W^{\lambda^{[k]}}(i_k + 1) : L^{\mu^{[k]}} \right] \end{aligned}$$

The lemma is proved □

We define v -decomposition numbers of $\mathcal{S}(\Lambda_{n_k})$ for $k = 1, \dots, g$ by

$$d_{\lambda^{[k]}\mu^{[k]}}(v) := \sum_{i_k \geq 0} \left[W^{\lambda^{[k]}}(i_k)/W^{\lambda^{[k]}}(i_k + 1) : L^{\mu^{[k]}} \right] \cdot v^i$$

as in the case of $\mathcal{S}(\Lambda)$. Then we have the following theorem.

Theorem 2.14. *For $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have*

$$d_{\lambda\mu}(v) = \overline{d}_{\lambda\mu}(v) = \prod_{k=1}^g d_{\lambda^{[k]}\mu^{[k]}}(v).$$

Proof. The first equality follows from Theorem 2.8. So we prove the second equality. By Lemma 2.13, we have

$$\begin{aligned}
\bar{d}_{\lambda\mu}(v) &= \sum_{i \geq 0} [\bar{Z}^\lambda(i) / \bar{Z}^\lambda(i+1) : \bar{L}^\mu] \cdot v^i \\
&= \sum_{i \geq 0} \left(\sum_{(i_1, \dots, i_g) \in \Delta_{i,g}} \prod_{k=1}^g [W^{\lambda^{[k]}}(i_k) / W^{\lambda^{[k]}}(i_k+1) : L^{\mu^{[k]}}] \right) \cdot v^i \\
&= \sum_{i \geq 0} \sum_{(i_1, \dots, i_g) \in \Delta_{i,g}} \left(\prod_{k=1}^g [W^{\lambda^{[k]}}(i_k) / W^{\lambda^{[k]}}(i_k+1) : L^{\mu^{[k]}}] \cdot v^{i_k} \right) \\
&= \sum_{(i_1, \dots, i_g) \in \mathbb{Z}_{\geq 0}^g} \left(\prod_{k=1}^g [W^{\lambda^{[k]}}(i_k) / W^{\lambda^{[k]}}(i_k+1) : L^{\mu^{[k]}}] \cdot v^{i_k} \right) \\
&= \prod_{k=1}^g \sum_{i_k \geq 0} [W^{\lambda^{[k]}}(i_k) / W^{\lambda^{[k]}}(i_k+1) : L^{\mu^{[k]}}] \cdot v^{i_k} \\
&= \prod_{k=1}^g d_{\lambda^{[k]}\mu^{[k]}}(v).
\end{aligned}$$

This proves the theorem. \square

3. v -DECOMPOSITION NUMBERS FOR ARIKI-KOIKE ALGEBRAS

We keep the notation in the previous section. We consider the v -decomposition numbers of the Ariki-Koike algebra \mathcal{H} , and show that similar results hold as in the previous section.

3.1. Let $\omega = (-, \dots, -, (1^n))$ be the r -partition and T^ω be the ω -tableau of type ω . Since $\varphi_{T^\omega T^\omega}$ is an identity map on M^ω and a zero map M^μ for $\mu \in \Lambda$ such that $\mu \neq \omega$, $\varphi_{T^\omega T^\omega}$ is an idempotent in \mathcal{S} . Moreover we see that $\varphi_{T^\omega T^\omega} \mathcal{S} \varphi_{T^\omega T^\omega} = \text{Hom}_{\mathcal{H}}(M^\omega, M^\omega) = \text{Hom}_{\mathcal{H}}(\mathcal{H}, \mathcal{H}) \cong \mathcal{H}$. It is well known that, for an \mathcal{S} -module M , $M\varphi_{T^\omega T^\omega}$ becomes a \mathcal{H} -module through the isomorphism $\varphi_{T^\omega T^\omega} \mathcal{S} \varphi_{T^\omega T^\omega} \cong \mathcal{H}$. Then we can define a functor, the so-called “Schur functor”, from the category of right \mathcal{S} -modules to the category of right \mathcal{H} -modules by $M \mapsto M\varphi_{T^\omega T^\omega}$. The following facts are known by [JM, Proposition 2.17].

$$(3.1.1) \quad W^\lambda \varphi_{T^\omega T^\omega} \cong S^\lambda \quad \text{as } \mathcal{H}\text{-modules} \quad (\lambda \in \Lambda^+)$$

$$(3.1.2) \quad L^\mu \varphi_{T^\omega T^\omega} \cong D^\mu \quad \text{as } \mathcal{H}\text{-modules} \quad (\mu \in \Lambda^+)$$

$$(3.1.3) \quad [W^\lambda : L^\mu] = [S^\lambda : D^\mu] \quad (\lambda, \mu \in \Lambda^+ \text{ such that } D^\mu \neq 0)$$

where $[S^\lambda : D^\mu]$ is the decomposition number of D^μ in S^λ .

3.2. One can define the Jantzen filtration of the Specht module S^λ in a similar way as in the case of W^λ , and we use a similar notation for this case. Then one can

define the v -decomposition number of \mathcal{H} , for $\lambda, \mu \in \Lambda^+$ such that $D^\mu \neq 0$, by

$$d_{\lambda\mu}^{\mathcal{H}}(v) := \sum_{i \geq 0} [S^\lambda(i)/S^\lambda(i+1) : D^\mu] \cdot v^i.$$

We have the following lemma.

Lemma 3.3. *Let $\lambda \in \Lambda^+$ and $i \geq 0$. Under the isomorphism in (3.1.1), we have*

$$W^\lambda(i)\varphi_{T^\omega T^\omega} = S^\lambda(i).$$

Proof. It is clear that $W^\lambda\varphi_{T^\omega T^\omega}$ has a basis $\{\varphi_T \mid T \in \mathcal{T}_0(\lambda, \omega)\}$. We have a bijective correspondence between $\mathcal{T}_0(\lambda, \omega)$ and $\text{Std}(\lambda)$ by $T \leftrightarrow \mathfrak{t}$ such that $\omega(\mathfrak{t}) = T$. Moreover, under the isomorphism in (3.1.1), we have

$$(3.3.1) \quad \varphi_T \varphi_{T^\omega T^\omega} = \begin{cases} m_{\mathfrak{t}} & \text{if } T \in \mathcal{T}_0(\lambda, \omega) \\ 0 & \text{if } T \notin \mathcal{T}_0(\lambda, \omega) \end{cases}$$

$$(3.3.2) \quad \langle \varphi_S, \varphi_T \rangle = \langle m_{\mathfrak{s}}, m_{\mathfrak{t}} \rangle_{\mathcal{H}} \quad \text{for } S = \omega(\mathfrak{s}), T = \omega(\mathfrak{t}) \in \mathcal{T}_0(\lambda, \omega)$$

by a similar argument as in the proof of [M2, Theorem 4.18].

First, we show the inclusion $W^\lambda(i)\varphi_{T^\omega T^\omega} \subseteq S^\lambda(i)$. Take $x \in W_R^\lambda(i)$. Then $\langle x, \varphi_T \rangle \in \wp^i$ for any $T \in \mathcal{T}_0(\lambda)$. It follows that

$$\langle x \cdot \varphi_{T^\omega T^\omega}, \varphi_T \rangle = \langle x, \varphi_T \cdot \varphi_{T^\omega T^\omega} \rangle \in \wp^i \quad \text{for any } T \in \mathcal{T}_0(\lambda).$$

This shows that

$$\langle x \cdot \varphi_{T^\omega T^\omega}, m_{\mathfrak{t}} \rangle_{\mathcal{H}} \in \wp^i \quad \text{for any } \mathfrak{t} \in \text{Std}(\lambda)$$

by (3.3.1) and (3.3.2). Hence $x \cdot \varphi_{T^\omega T^\omega} \in S_R^\lambda(i)$, and the claim follows by taking the quotient.

Next, we show the converse inclusion $W^\lambda(i)\varphi_{T^\omega T^\omega} \supseteq S^\lambda(i)$. Take $y \in S^\lambda(i)$. Then we have

$$(3.3.3) \quad \langle y, m_{\mathfrak{s}} \rangle_{\mathcal{H}} \in \wp^i \quad \text{for any } \mathfrak{s} \in \text{Std}(\lambda).$$

Write $y = \sum_{\mathfrak{t} \in \text{Std}(\lambda)} r_{\mathfrak{t}} m_{\mathfrak{t}}$, and put $x = \sum_{T \in \mathcal{T}_0(\lambda, \omega)} r_{\mathfrak{t}} \varphi_T \in W^\lambda$, where T is the λ -tableau of type ω corresponding to \mathfrak{t} . Then we have $y = x \cdot \varphi_{T^\omega T^\omega}$, and

$$\langle x, \varphi_S \rangle \in \wp^i \quad \text{for any } S \in \mathcal{T}_0(\lambda, \omega)$$

by (3.3.1), (3.3.2) and (3.3.3). Since $\langle \varphi_T, \varphi_S \rangle = 0$ if the type of T is not the same as the type of S , we have

$$\langle x, \varphi_S \rangle \in \wp^i \quad \text{for any } S \in \mathcal{T}_0(\lambda).$$

This shows that $x \in W_R^\lambda(i)$, and the claim follows. The lemma is proved. \square

This lemma implies the following proposition.

Proposition 3.4. *Take $\lambda, \mu \in \Lambda^+$ such that $D^\mu \neq 0$. Then for any $i \geq 0$, we have*

$$[W^\lambda(i)/W^\lambda(i+1) : L^\mu] = [S^\lambda(i)/S^\lambda(i+1) : D^\mu].$$

In particular, we have $d_{\lambda\mu}(v) = d_{\lambda\mu}^{\mathcal{H}}(v)$.

Proof. We consider the \mathcal{S} -module filtration

$$W^\lambda(i) = W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_k = W^\lambda(i+1)$$

such that $W_j/W_{j+1} \cong L^{\mu_j}$. By applying the Schur functor, together with Lemma 3.3, we have

$$S^\lambda(i) = W^\lambda(i)\varphi_{T^\omega T^\omega} \supset W_1\varphi_{T^\omega T^\omega} \supset \cdots \supset W_k\varphi_{T^\omega T^\omega} = W^\lambda(i+1)\varphi_{T^\omega T^\omega} = S^\lambda(i+1),$$

where $W_j\varphi_{T^\omega T^\omega}/W_{j+1}\varphi_{T^\omega T^\omega} \cong (W_j/W_{j+1})\varphi_{T^\omega T^\omega} \cong L^{\mu_j}\varphi_{T^\omega T^\omega} \cong D^{\mu_j}$ by (3.1.2). The proposition follows from this. \square

For $\lambda \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = (n_1, \dots, n_g)$, $\lambda^{[k]}$ is an r_k -partition of n_k . Then we have the Specht module $S^{\lambda^{[k]}}$ and its unique quotient $D^{\lambda^{[k]}}$ for the Ariki-Koike algebra \mathcal{H}_{n_k, r_k} . Moreover for $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu) = (n_1, \dots, n_g)$, we have the v -decomposition number $d_{\lambda^{[k]}\mu^{[k]}}^{\mathcal{H}}(v)$ for $\mathcal{H}_{n_k r_k}$. Combining Theorem 2.14 with Proposition 3.4, we have the following result.

Theorem 3.5. *Let $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$. Assume that $D^\mu \neq 0$ and $D^{\mu^{[k]}} \neq 0$ for any $k = 1, \dots, g$. Then we have*

$$d_{\lambda\mu}^{\mathcal{H}}(v) = \prod_{k=1}^g d_{\lambda^{[k]}\mu^{[k]}}^{\mathcal{H}}(v).$$

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